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On the θ-theories and the multivalued wave functions

SŁAWOMIR KLIMEK-CHUDY

Department of Mathematical
Methods in Physics
University of Warsaw
Hoza 74, Warsaw, Poland

WITOLD III KONDRACKI

Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8, Warsaw, Poland

Abstract. *The interpretation of θ-theories in terms of multivalued wave functions is given.*

The notion of θ-vacua was introduced by Callan, Dashen, Gross [1] and Jackiw, Rebbi [2] in 1976 for the case of Yang-Mills field theory. In 1978 Singer [3] pointed out that the configuration space of Yang-Mills field theory is multiply connected. Then Gawedzki [4], Dowker [5] and Isham [6] showed that θ-vacua are closely related to the multiple connectivity of the configuration space in that theory. Their arguments coincide with those contained in [7, 8, 9, 10] and concern the problem of a formulation of Quantum Mechanics on multiply-connected regions. In those papers two approaches were presented: the Feynman path integral quantization and the representation of the Canonical Commutation Relations (C.C.R.).

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In this paper we give the interpretation of θ-theories in terms of multivalued wave functions. The wave function Ψ in quantum mechanics is not physicaly measurable so there are no reasons for its single-valuedness. On the other hand the observables are measurable so they are single valued. We formulate postulates to be fulfilled by a multi-valued ψ such that the physical quantities (observables) computed in this state be single valued. We show (Theorems 1, 2) that if the configuration space M is multiply-connected then multivalued wave functions are characterized by characters of $\Pi_1(M)$. Whenever $\Pi_1(M) = 0$, the multivalued wave functions reduce to ordinary single valued functions. Theorems 3, 4 gives the correspondence and interpretation of our approach in terms of C.C.R. The characters of $\Pi_1(M)$ denoted by θ are identified with those introduced in ref. [1, 2] and can be interpreted as strictly conserved quantum quantities. There are given examples that show the physical situations where θ-vacua appears.

1. θ-THEORIES

Let M be a connected manifold interpreted as a configuration space of the system, and (P, Π, M) be the universal covering of M. We consider a subset $\Psi \subset M \times \mathbb{C}^1$ with the following properties:

1) $pr_M \Psi = M$, where pr_M is the natural projection onto the first factor

2) $\forall x \in M$, $\exists$ an open neighbourhood U of x in M such that

$$\Psi \cap pr_M^{-1}(U) = \bigcup_{i \in I} \psi_{i,u}$$

where $\psi_{i,u}$ are the graphs of $\mathbb{C}^1$-functions on U and I is a finite or countable index

3) $\forall i,j \in I \quad \psi^*_{i,u} \ \psi_{i,u} = \psi^*_{j,u} \ \psi_{j,u}$

4) $\forall i,j \in I \quad \psi^*_{i,u} \ \psi_{i,u} = \psi_{j,u} X \psi_{j,u}$

where X is an arbitrary vector field on U

5) Ψ is connected.

The open subsets appearing in the postulate 2) are called the proper subsets, and functions $\psi_{i,u}$-pieces over U. Ψ-fulfilling 1), 2) will be called the graph of a multivalued function, or simply a multivalued function. Postulates 3), 4) assure the possibility of defining expectation values of positions and momenta. The requirement 5) reduces multivalued function to a single valued one if the multivaluedness is not physical.

If the properties 1) - 5) are satisfied, then Ψ will be called a multivalued wave function. The following theorems characterize multivalued wave functions on M

THEOREM 1. *Let Ψ be a multivalued wave function such that for each proper subset U and $\forall_j \in I$, $i \neq j$.*

$$(*) \qquad \psi_{i,u} \cap \psi_{j,u} = \phi.$$

Then there exist - a) a character θ of $\pi_1(M)$

b) a C^1-function $\hat{\psi} : P \longrightarrow \mathbb{C}^1$, non-vanishing and «quasiperiodic» with respect to θ i.e.

$$\forall p \in P, \quad \forall [w] \in \Pi_1(M)$$

$$\hat{\psi}(p \cdot [w]) = \theta([w])\, \hat{\psi}(p)$$

where « . » means the natural right action of $\Pi_1(M)$ on P such that $\forall x \in M$ $\Psi(x) = \bigcup_{p \in \pi^{-1}(x)} \hat{\psi}(P)$.

THEOREM 2. *Let θ-be a character of $\Pi_1(M)$ and $\hat{\psi} : P \longrightarrow \mathbb{C}^1$ be a C^1-function quasiperiodic with respect to θ. Then, the function $\Psi(x) = \bigcup_{p \in \pi^{-1}(x)} \psi(p)$ is a multivalued wave function.*

Let us note that the condition (*) together with the postulate 3) imply that ψ is non-vanishing if I contains more than one element. This seems to be not physical.

Moreover one can easily construct a multivalued wave function Ψ which doesn't define any quasiperiodic $\hat{\psi}$. On the other hand we do not need to assume that $\hat{\psi}(x) \neq 0$ in any point x in Theorem 2 and nevertheless the multivalued function determined by $\hat{\psi}$ in this way is still a multivalued wave function. This observation leads us to the following conclusion: all physically relevant multivalued wave functions are of the type described by Theorem 2. In other words the true wave functions in quantum mechanics are quasiperiodic functions on the universal covering of M. This statement will be demonstrated in examples.

Proof of Theorem 1 (construction). Let $x_0 \in M$ be an arbitrary point. We have $\Psi(x_0) = \bigcup_{i \in I} \psi_{i,u}(x_0)$ and let us choose an arbitrary piece $\psi_{0,u}$. In this proof the index u will be omitted.

LEMMA 1. *Let $w : [0,1] \longrightarrow M$, $w(0) = x_0$ be a path starting at x_0 and let ψ satisfy the assumption of Theorem 1. Then there exists a unique lifting $\tilde{w}_{\psi_0} : [0,1] \longrightarrow \Psi$ such that*

$$\tilde{w}_{\psi_0}(0) = \psi_0(x_0)$$

$$pr_M \tilde{w}_{\psi_0} = w.$$

Proof: Case 1 - Let the image of w be contained in a proper subset. Then $\tilde{w}_{\psi_0} := \psi_0 w$ is the required lifting.

Case 2 (general) - Using the compactness of [0, 1] and the definition of multivalued function we can choose $t_0, \ldots, t_n$ $(0 = t_0 < t_1 < \ldots < t_n = 1)$ such that for every i $w([t_i, t_{i+1}])$ is contained in a proper subset. Now we can lift w step by step using Case 1.

The uniqueness follows from the fact that [0, 1] is connected. As a matter of fact, let $\tilde{w}^1_{\psi_0}, \tilde{w}^2_{\psi_0}$ be two different lifting. We define

$$A = \{t \in [0, 1] : \tilde{w}^1_{\psi_0}(t) = \tilde{w}^2_{\psi_0}(t)\}$$

$$B = \{t \in [0, 1] : \tilde{w}^1_{\psi}(t) \neq \tilde{w}^2_{\psi_0}(t)\}$$

$$A \cap B = \phi, \quad A \cup B = [0, 1], \quad 0 \in A.$$

We will show that A and B are open subsets in [0, 1] therefore $B = \phi$. Let $t \in [0, 1]$ and U - be a proper neighbourhood of $w(t)$. If $t \in A$ then $\tilde{w}^1_{\psi_0}(t) = \tilde{w}^2_{\psi_0}(t)$ belongs to a certain piece over U. Note that $\tilde{w}^1_{\psi_0}$ is equal to $\tilde{w}^2_{\psi_0}$ over U, because different pieces belong by virtue of the assumption (*), to different arcwise components of $\Psi \cap pr_M^{-1}(U)$. Therefore $w^{-1}(u)$ is the open neighbourhood of t contained in A. By the same argument B is open. ■

LEMMA 2. *Let* $w : [0, 1] \longrightarrow M$, $w(0) = x_0$ *- be a path starting at* x_0 *and let* Ψ *satisfy the assumptions of Theorems 1. Then for every homotopy* $\Omega : [0, 1] \times [0, 1] \longrightarrow M$ *such that* $\Omega(t, 0) = w(t)$ *there exists a unique lifting* $\tilde{\Omega} : [0, 1] \times [0, 1] \longrightarrow \psi$ *such that*

$$\tilde{\Omega}(t, 0) = \tilde{w}_{\psi_0}(t).$$

The proof is analogous to that of Lemma 1. ■

COROLLARY 1. *Let* Ψ *be as in Theorem 1. If* w^1, w^2 *are two paths in* M *starting at* x_0 *and ending at* x, *and if there exists a homotopy* Ω_t *such* $\Omega_0 = w^1$, $\Omega_1 = w^2$ *and for every* t, $\Omega_t(0) = x_0$, $\Omega_t(1) = x$ *(we write* $w^1 \sim w^2 \operatorname{rel} 0, 1$ *), then* $\tilde{w}^1_{\psi_0} \sim \tilde{w}^2_{\psi_0} \operatorname{rel}\{0, 1\}$ *and particularly*

$$\tilde{w}^1_{\psi_0}(1) = \tilde{w}^2_{\psi_0}(1).$$ ■

Let us note that $\Pi_1(M)$ acts transitively in $\Psi(x)$ for any x. If $\psi_i(x)$, $\psi_j(x) \in \Psi(x)$ then there exists a path $\tilde{w} : [0, 1] \longrightarrow \Psi$ such that $\tilde{w}(0) = \psi_i(x)$, $\tilde{w}(1) = \psi_j(x)$ (postulate 5). Then the needed element of $\pi_1(M)$ is $[pr_M \tilde{w}]$.

Those statements are quite analogous to those in the theory of covering spaces [11].

Let us recall the definition of the universal covering space which is the most adequate for our purposes.

The total space P of the universal covering of M is the set of all homotopy classes of paths on M starting from x_0 rel $\{0, 1\}$. The fiber over x consists of homotopy classes of paths from x_0 to x rel $\{0, 1\}$. Let $P \in p = [w]$. We put

$$\hat{\psi}(p) = \tilde{w}_{\psi_0}(1).$$

By virtue of Corollary 1, this definition does not depend on the choice of $w \in [w]$.

Now let $[w] \in \Pi_1(M, x_0)$. Let us define

$$\theta_{x_0, \psi_0} : \pi_1(M, x_0) \ni [w] \longrightarrow \tilde{w}_{\psi_0}(1)\, [\psi_0(x_0)]^{-1} \in S^1.$$

The inclusion followes from postulate 3. Now we prove that θ_{x_0, ψ_0} does not depend on the choice of ψ_0. Let us take the function:

$$\gamma_{i, w} : [0, 1] \ni t \longrightarrow \tilde{w}_{\psi_i}(t)\, [\tilde{w}_{\psi_0}(t)]^{-1} \in S^1 \quad [w] \in \Pi_1(M).$$

Using the compactness of $[0, 1]$ we can choose $t_0, \ldots, t_n$ $(0 = t_0 < t_1 < \ldots < t_n = 1)$ such that $w([t_j, t_{j+1}])$ in contained on a proper subset. The Requirement 4) shows, that $\gamma_{i,w}$ is constant on every $[t_j, t_{j+1}]$, hence it is constant on $[0, 1]$. In particular

$$\tilde{w}_{\psi_i}(1)\, [\psi_i(x_0)]^{-1} = \tilde{w}_{\psi_0}(1)\, [\psi_0(x_0)]^{-1} = \theta_{x_0, \psi_0}([w])$$

so θ_{x_0}, ψ_0; is ψ_0-independent.

We conclude by means of the same arguments for $w : [0, 1] \longrightarrow M$ such that $w(0) = x_0$, $w(1) = x$, that for a given multivalued wave function Ψ satysfying the assumption of Theorem 1, θ is a perfectly well defined character of $\Pi_1(M)$. As a matter of fact, let w_1, w_2- be two loop at x_0. Then

$$\begin{aligned} \theta([w^1][w^2]) &= (w^1 0 w^2)^{\sim}_{\psi_0}(1)[\psi_0(x_0)]^{-1} = \\ &= \tilde{w}^1_{\tilde{w}^2_{\psi_0}(1)}(1)[\psi_0(x_0)]^{-1} = \tilde{w}^1_{\tilde{w}^2_{\psi_0}(1)}(1)[\tilde{w}^2_{\psi_0}(1)]^{-1} \times \\ &\times [\tilde{w}^2_{\psi_0}(1)][\psi_0(x_0)]^{-1} = \theta([w^1])\, \theta([w^2]). \end{aligned}$$

The quasiperiodicity of $\hat{\psi}$ follows from simple calculations. Let $p = [\delta]$ and $[w] \in \Pi_1(M, x_0)$. Then

$$\begin{aligned} \hat{\psi}(p \cdot [w]) &= \psi([\delta\, w]) = (\delta\, w)^{\sim}_{\psi_0}(1) = \tilde{\delta}_{\tilde{w}_{\psi_0}(1)}(1) = \\ &= \tilde{w}_{\tilde{\psi}_0(1)}[\psi_0(x_0)]^{-1}\delta_{\psi_0}(1) = \theta([w])\, \psi(p). \end{aligned}$$

■

REMARK 1. It we choose a piece ψ_i instead of ψ_0, then the obtained function $\hat{\psi}$ shall differ by a constant $U(1)$-factor which does not effect quantum mechanical effects.

REMARK 2. $\hat{\Psi}$ is of class C^1 since Ψ_i are.

Proof of Theorem 2. Let us define $\Psi \subset M \times \mathbb{C}^1$ by the formula:

$$\Psi(x) = \bigcup_{p \in \Pi^{-1}(x)} \hat{\Psi}(p)$$

Then direct computation shows that the postulates 1) to 5) hold. ■

Now we give another representation for multivalued wave functions. The aim of the following theorem will be to describe the state by a single valued function. This can be obtained by a suitable redefinition of the momentum operator. However, in that way we would loose the uniqueness of such a wave function.

Let $u_{1,2} : M \longrightarrow \mathbb{C}^1$ be C^1-functions on M and $\Omega_{1,2}$ be closed 1-forms on M. We say that the pair (u_1, Ω_1) is equivalent to the pair (u_2, Ω_2) if

$$1^\circ \ \Omega_1 - \Omega_2 \in H^1(M, Z^1)$$

$$2^\circ \ u_2(x) = u_1(x) \exp\left(2\Pi i \int_w (\Omega_1 - \Omega_2)\right)$$

where w is a path joining x_0 and x.

The above definition does not depend on the choice of w. We will use the abbreviation $[u, \Omega]$ for equivalence classes of this relation. u will be called the Bloch factor and Ω-the magnetic form. This terminology emphazises the connection with the Bloch theorem in solid state physics [12].

THEOREM 3. *There is a bijection between the set of quasiperiodic (with respect to some character θ of $\Pi_1(M)$) functions $\hat{\psi} : P \longrightarrow \mathbb{C}^1$ and the set of classes $[u, \Omega]$.*

Moreover

a) $u^*u = \hat{\psi}^*\hat{\psi}$

b) $u^*(X + 2\Pi i \langle X, \Omega\rangle)u = \hat{\psi}^*\hat{X}\hat{\psi}$

where X - a vector field on M

$\hat{X}$ - the (unique) lift of X to P.

Proof. We put

$$\theta([w]) = e^{2\Pi i \int_w \Omega}$$

where $[w] \in \Pi_1(M)$, $w \in [w]$, $(u, \Omega) \in [u, \Omega]$. Then,
θ - is independent on the choice of w, since Ω is closed
θ - is independent on the choice of Ω
θ - is a character because

$$\theta([w_1][w_2]) = e^{2\Pi i \int_{w_1 \cdot w_2} \Omega} = e^{2\Pi i \int_{w_1} \Omega} e^{2\Pi i \int_{w_2} \Omega} = \theta([w_1])\theta([w_2]).$$

Let $P \ni p = [\delta]$, where $[\delta]$ is a homotopy class of pahts from x_0 to $\Pi(p)$.
We put

$$\hat{\psi}(p) := e^{2\pi i \int_\delta \Omega} u(\Pi(p)). \quad \text{Then}$$

$\hat{\psi}$ - is independent on the choice of $\delta \in [\delta]$
$\hat{\psi}$ - is independent on the choice of $(u, \Omega) \in [u, \Omega]$ because

$$e^{2\Pi i \int_\delta \Omega_1} u_1(\Pi(p)) = e^{2\Pi i \int_\delta \Omega_2} e^{2\Pi i \int_\delta \Omega_1 - \Omega_2} u_1(\Pi(p)) =$$
$$= e^{2\Pi i \int_\delta \Omega_2} u_2(\Pi(p))$$

$\hat{\psi}$ - is quasiperiodic because

$$\hat{\psi}(p[w]) = e^{2\Pi i \int_{\delta w} \Omega} u(\Pi(p)) = e^{2\Pi i \int_w \Omega} e^{2\Pi i \int_\delta \Omega} u(\pi(p)) =$$
$$= \theta([w])\, \hat{\psi}(p).$$

Thus we have proved 1°. Instead of performing similar calculation to prove 2° we refer to Theorem of Behnke-Stein [13] which in fact contains the same ideas. The prove of the equalities a), b) is direct. ■

To complete the discusion of θ-theories we shall describe the approach based on line bundles and flat connections considered in [6, 14]. When the configuration space is multiply-connected then inequivalent realisations of the C.C.R. are labeled by characters of $\Pi_1(M)$. Moreover they can be explicity constructed in terms of geometrical objects. The following theorem gives us the formal correspondence with our approach. Let
E_1, E_2 - be trivial, Hermitian complex line bundles
α_1, α_2 - be flat connections on E_1, E_2 preserving Hermitian structures,
s_1, s_2 - be sections of E_1, E_2.

The triplet (E_1, α_1, s_1) will be called equivalent to (E_2, α_2, s_2), if there exists an isomorphism ϕ of bundles E_1, E_2 such that

$$\alpha_1 = \phi^* \alpha_2, \quad s_1 = \phi \cdot s_2.$$

The classes of the above equivalence relation are denoted by $[E, \alpha, s]$

THEOREM 4.

1° *The class* $[E, \alpha, s]$ *defines a character* θ *of* $\Pi_1(M)$ *and a quasiperiodic (with respect to* θ*) function* $\hat{\Psi} : P \longrightarrow \mathbb{C}^1$

2° *A quasiperiodic (with respect to some character* θ *of* $\Pi_1(M)$*) function* $\hat{\psi} : P \longrightarrow \mathbb{C}^1$ *defines a class* $[E, \alpha, s]$.

Moreover, if $[E, \alpha, s]$ *corresponds to* $\hat{\psi}$ *then*

$$(s, s) = \hat{\psi}^* \hat{\psi}$$

$(s, \alpha_X s) = \hat{\psi}^* X \hat{\psi}$ *where* X *is a vector field on* M *and* $\hat{X}$ *its lifting to* P.

Proof. Let $(E, \alpha, s) \in [E, \alpha, s]$ and let (P_E, Π, M) denote the principal S^1 bundle of orthogonal frames for (E, α, s). Then E is an associated vector bundle of (P_E, Π, M). Let us note that

a) Every section s of E defines an equivariant function $K(s) : P_E \longrightarrow \mathbb{C}^1$ such that

$$K(s)(p \cdot g) = g\, K(s) p \quad \text{where} \quad p \in P_E, \quad g \in S^1.$$

Moreover the map K is bijective

b) For any connection α preserving hermitian structure of E there exists a connection A on (P_E, Π, M) such that

$$\alpha_X s = K^{-1} \hat{X}\, K(s)$$

where $\hat{X}$ is the horizontal lifting of X with respect to A. The converse is also true [15].

Let us choose $x_0, p_0 \in P_E$, $\Pi(p_0) = x_0$ and let $\delta : [0, 1] \longrightarrow M$ be a path such that $\delta(0) = x_0$. The homotopy class of δ is an element of P. We put

$$\hat{\psi}([\delta]) := K(s)(\hat{\delta}_{p_0}(1))$$

where $\hat{\delta}_{p_0}$ is the horizontal lifting of δ to P_E with respect to A, starting at p_0.

This definition is obviously independent on the choice of $(E, \alpha, s) \in [E, \alpha, s]$. We shall prove that $\hat{\Psi}$ is quasiperiodic. For any $w : [0, 1] \longrightarrow M$ such that $w(0) = w(1) = x_0$

$$\theta([w]) := \hat{w}_{p_0}(1)/p_0.$$

It is an element of the holonomy group of A. We have

$$\hat{\psi}([\delta][w]) = K(s)(\delta w_{p_0}(1)) = K(s)(\hat{\delta}_{\hat{w}_{p_0}(1)}(1)) =$$

$$= K(s)(\hat{\delta}_{p_0}(1) w_{p_0}(1)/p_0) = w_{p_0}(1)/p_0 K(s)(\delta_{p_0}(1)) =$$

$$= \theta([w])\, \hat{\psi}([\delta]).$$

This end the proof of part 1. In order to prove the conserve statement let us take a quasiperiodic function $\hat{\psi} : P \longrightarrow \mathbb{C}^1$.

Let $E = PX_\theta \mathbb{C}^1$ be the line bundle associated to (P, Π, M) with respect to the representation ρ_θ of $\Pi_1(M)$ in $\mathbb{C}^1$ given by the formula:

$$\rho_\theta[w]\xi = \theta([w])\xi \quad [w] \in \Pi_1(M) \quad \xi \in \mathbb{C}^1.$$

Let us note that (P, Π, M) as a principal bundle has an unique connection A and moreover the quasiperiodic function $\hat{\psi}$ is equivariant in the sense of remark a). Thus, according to the remarks a), b) there exist a natural connection α, and a natural section s in E. It is easy to see, that the quasiperiodic function on P determined through the construction of part 1 by triplet (E, α, s) is just $\hat{\psi}$. Now let (E', α', s) define the same $\hat{\psi}$ as (E, α, s). For the proof that (E, α, s) is isomorphic to (E', α', s') see [16, 17]. ■

The above considerations have strictly topological character. However, to discuss the influence of the multiple connectivity onto quantum mechanics in more details it is necessary to introduce additional structures on M i.e. a riemanian metric and a measure. This gives the possibility to investigate the Hilbert space of states, the Hamiltonian operator and the algebra of observables. There are several ways to construct the Hilbert space and the Hamiltonian operator according to the representation theorems. We will discuss it in the next publication. However in any case θ has clear physical interpretation: θ is strictly conserved quantum quantity. We would like to point that:

- θ is a mathematical object rather than a number
- θ is a in principle measurable
- The nature of θ is strictly topological.

2. EXAMPLES

2.1. Rotor (pendulum) [4, 18, 19, 20]. This is the simplest, academic example in which θ appears. The Hamiltonian for this problem is

$$H = -\frac{1}{2}\frac{d^2}{d\phi^2}\ (+\cos\phi)$$

where ϕ is the angle variable.

Thus the configuration space $M = S^1$ and its universal covering $P = R^1$. Since $\Pi_1(S_1) = Z$ then the set of characters is S^1. By general prescription we are looking for the eigenfunction in the form

$$H\hat{\psi}_\theta = E_\theta \hat{\psi}_\theta \quad \text{where}$$

$$\hat{\psi}_\theta : R^1 \longrightarrow \mathbb{C}^1 \quad \text{such that} \quad \hat{\psi}_\theta(x + 2\Pi) = \theta\, \hat{\psi}_\theta(x) \quad \theta \in S^1$$

θ has an influence on the eigenvalues of the Hamiltonian

2.2. The Aharonov-Bohm experiment [21, 22]. Let us consider an infinite, cylindric solenoid ot small radius. The magnetic field vanishes outside the cylinder. One can easily check that the wave function of an electron in the presence of the solenoid is given by

$$\psi = e^{i\theta\phi}\psi_0$$

where ϕ is a polar angle in the perpendicular plane of the cylinder.
ψ_0 is a single valued solution of the free Schrödinger equation
θ is some constant depending on the radius of solenoid and the magnetude of the current.

We see that ψ is multivalued in accordance to the theory because the configuration space (the space outside the selenoid) is multiply connected. Let us notice that θ (mod 2Π) i.e. the character, can be measured in the quantum mechanical way. Thus it is a quantum number.

2.3. Periodic crystal [12]. The configuration space M for an electron in the periodic crystal is 3-dim torus T^3. We refer to the Bloch theorem which summerized the θ-theory in this case. We mention only that in the terminology of solid state physics the space of characters of $\Pi_1(M)$ is called the first Brillouin zone and the elements of this space i.e. characters are called quasi-momenta.

2.4. Yang-Mills field theory over S^3, T^3 [3, 14, 23]. The configuration space for Yang-Mills field theory was introduced by Singer as the space of gauge equivalent connections. One can compute (see for example [24]) that $\Pi_1(M) = Z$ for S^3 and $\Pi_1(M) = Z \times \Pi_1(G) \times \Pi_1(G) \times \Pi_1(G)$ for T^3 where G is the structural group of this theory. In the case when the space-like surface is S^3 the space of characters is S^1. These are the original θ-numbers of [1, 2] (see also [6]). For the T^3 case the part of characters connected with $\Pi_1(G) \times \Pi_1(G) \times \Pi_1(G)$ are the electric fluxes of 't Hooft. The physical interpretation of these quantum numbers can be find in [25, 26, 27].

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REFERENCES

[1] C.G. CALLAN, R.F. DASHEN, D.J. GROSS: *The structure of the gauge theory vacuum*, Phys. Lett. B 63, 334 - 341, (1976).
[2] R. JACKIW, C. REBBI: *Vacuum periodicity in a Yang-Mills Quantum Theory*. Phys. Rev. Lett. 37, 172 - 172, (1976).
[3] I.M. SINGER: *Some remarks on the Gribov ambiguity*, Comm. Math. Phys. 60, 7 - 13, (1978).
[4] K. GAWEDZKI - unpublished.
[5] J.S. DOWKER - Austin preprint (1980).
[6] C.J. ISHAM - *Topological θ-sector in canonically quantized gravity*, Phys. Lett. B 106, 188 - 197, (1981).
[7] A.S. WIGHTMAN in *Theoretical Physics*, D. Feldman ed. Benjamin, New York (1967).
[8] L.S. SCHULMAN: *Approximate topologies*, J. Math. Phys. 12, 304 - 308, (1971).
[9] M. LAIDLAW, C. DEWITT: *Feynman functional integrals for systems of indistinguishable particles*, Phys. Rev. D3, 1375 - 1378, (1971).
[10] J.S. DOWKER: *Quantum Mechanics and Field Theory on multiply connected and on homogeneous spaces*, J. Phys. A5, 936 - 943, (1972).
[11] M.J. GREENBERG: *Lectures on Algebraic Topology*, Benjamin, New York (1976).
[12] N. ASHCROFT, D. MERMIN: *Solid State Physics*, Holt, Rinehard and Winston, New York (1976).
[13] O. FORSTER: *Riemannscher Flächen*, Springer Verlag, Berlin - Heidelberg - New York, (1977).
[14] M. ASOREY, P.K. MITTER - CERN, LPTHE preprint (1982).
[15] S. KOBAYASKI, K. NOMIZU: *Foundations of differential geometry*, Interscience Publishers, New York, (1963).
[16] B. KONSTANT in *Lecture Notes in Mathematics*, 170 Springer Verlag, Berlin - Heidelberg - New York (1970).
[17] M. ASOREY: *Some remarks on the classical vacuum structure of gauge field theories*, J. Math. Phys. 22, 179 - 185, (1981).
[18] K.D. ROTHE, J.A. SWIECA: *Quasi periodic boundary conditions and the vacuum structure in gauge theories*, Nucl. Phys. B 138, 26 - 31, (1978).
[19] C.W. BERNARD, E.J. WEINBERG: *Interpretation of pseudoparticles in physical gauges*, Phys. Rev. D15, 3653 - 3659, (1977).
[20] M. ASOREY, J.G. ESTEVE, A.F. PACHECO: *Planar rotor: The θ-vacuum structure and some approximate methods in quantum mechanics*, Phys. Rev. D27, 1852 - 1860, (1983).
[21] Y. AHARONOV, D. BOHM: *Significence of electromagnetic potentials in the Quantum Theory*, Phys. Rev. 115, 489 - 498, (1959).
[22] Y. AHARONOV, D. BOHM: *Further considerations on electromagnetic potentials in the Quantum Theory*, Phys. Rev. 123, 1511 - 1520, (1961).
[23] I.M. SINGER: *The geometry of the orbit space for non-abelian gauge theories*, Phys. Scripta 24, 817 - 821, (1981).

[24] S. KLIMEK, W. KONDRACKI: *The topology of the Yang-Mills theory over torus*, Proc. 12 th Winter School of Abstract Analysis (1984).

[25] G. 't HOOFT: *A property of electric and magnetic flux in non-abelian gauge theories*, Nucl. Phys. B 153, 141 - 164, (1978).

[26] G. 't HOOFT: *Aspects of quark confinement*, Phys. Scripta 24, 841 - 847, (1981).

[27] G. 't HOOFT: *Confinement and topology in non-abelian gauge theories*, Acta. Phys. Austr. Suppl. XXII, 531 - 574, (1980).

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